Approximation of Stopped Diffusion Processes

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Problem

$(X_t)_{t\geq 0}$: d-dimensional diffusion process :

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \qquad \text{(SDE)}$$

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- W d-dimensional Brownian motion,
- b, σ Lipschitz continuous in space, locally bounded in time.

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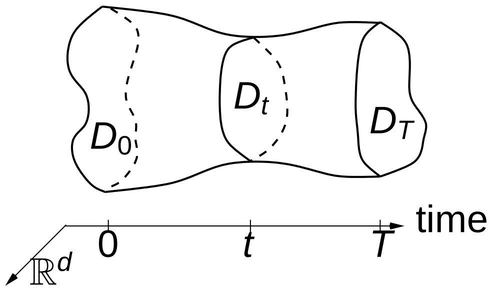

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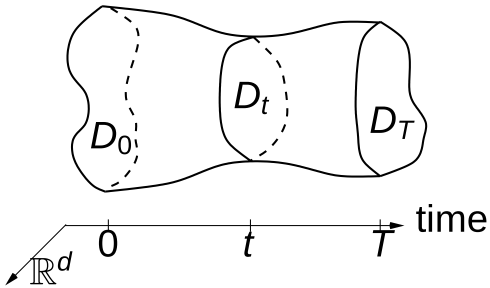

FIG.: Time-space domain and its time sections.

- $\tau := \inf\{t > 0 : X_t \notin D_t\}$.
- $\tau \wedge T$ exit time of $(t, X_t)_{t\geq 0}$ from the time-space domain $\mathcal{D}$.

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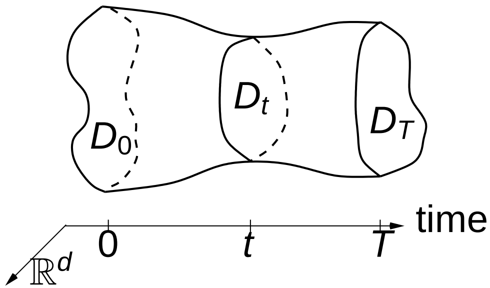

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Approximation of quantities of the following type

$$Q(T, g, f, k, x) := \mathbb{E}_x[g(\tau \wedge T, X_{\tau\wedge T})Z_{\tau\wedge T} + \int_0^{\tau\wedge T} Z_s f(s, X_s)ds], Z_s = \exp(-\int_0^s k(r, X_r)dr). \qquad \text{(Q)}$$

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Two Approximations needed

- Discretization of the process $(X_t)_{t\geq 0}$.
- Discretization of the stopping time $\tau\wedge T$.

The Black & Scholes framework

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\begin{aligned}
e^{rT}Q &:= \mathbb{P}[W_T \le \frac{\log(K/x_0)}{\sigma}, \sup_{s\in[0,T]} W_s \ge \frac{\log(H/x_0)}{\sigma}] \\
&= \mathbb{P}[W_1 \le c, \sup_{s\in[0,1]} W_s \ge b],\ c := \frac{\log(K/x_0)}{\sigma\sqrt{T}},\ b := \frac{\log(H/x_0)}{\sigma\sqrt{T}} \\
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$$\begin{aligned} e^{rT}Q^h &= \mathbb{P}[\bar{X}_m \le c\sqrt{m}, \bar{\tau} \le m], \bar{X}_j := \sum_{i=1}^{j} G_i, (G_i)_{i\ge 0}, \text{ i.i.d } \mathcal{N}(0,1), \bar{\tau} := \inf\{i \in \mathbb{N} : \bar{X}_i \ge b\sqrt{m}\} \\ &= \mathbb{P}[\bar{X}_m \ge 2(b\sqrt{m} + \bar{R}_m) - c\sqrt{m}], \bar{R}_m := \bar{X}_{\bar{\tau}} - b\sqrt{m} \text{ (Overshoot!)} \\ &\simeq (1 - \mathbb{E}[\Phi(2b + \frac{\bar{R}_m}{\sqrt{m}}) - c)]) \text{ (Asymptotic independence of } \bar{X}_m \text{ and } \bar{R}_m). \end{aligned}$$

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$$e^{rT}(Q^h - Q) \simeq -\mathbb{E}[\Phi(2(b + \frac{\bar{R}_m}{\sqrt{m}}) - c)] + \Phi(2b - c) \simeq -2\frac{E[\bar{R}_m]}{\sqrt{m}}\varphi(2b - c) + o(\frac{1}{\sqrt{m}}).$$

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- Asymptotics of the overshoot : $\mathbb{E}[\bar{R}_m] \underset{m}{\to} c_0 := \frac{\mathbb{E}[\bar{X}^2_{\bar{\tau}_+}]}{2\mathbb{E}[\bar{X}_{\bar{\tau}_+}]}$, $\bar{\tau}_+ := \inf\{i \ge 0 : \bar{X}_i > 0\}$ (Siegmund, Journal Appl. Proba. (79), Sigmund and Yuh, PTRF (82) $\rightsquigarrow$ renewal arguments, Borovkov, Sib. Math. Journal (62), Peres *et al*, AOP (97) $\rightsquigarrow$ complex analysis).

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- Correction Procedure : (Broadie, Glasserman, Kou, Math. Finance (97))

$$\tilde{Q}^h := e^{-rT}P[X_T \le K, \max_{s\in[0,T]} X_s \ge H\exp(-c_0\sigma\sqrt{h})],\ e^{rT}(\tilde{Q}^h - Q) = o\left(\frac{1}{\sqrt{m}}\right).$$

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Goal

- Characterize the overshoot for the Euler scheme of diffusion processes in time dependent domains.
- Establish an error expansion and a domain correction procedure in that framework.

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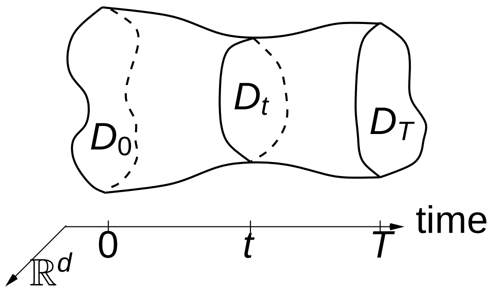

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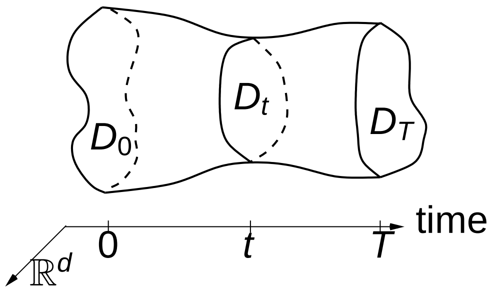

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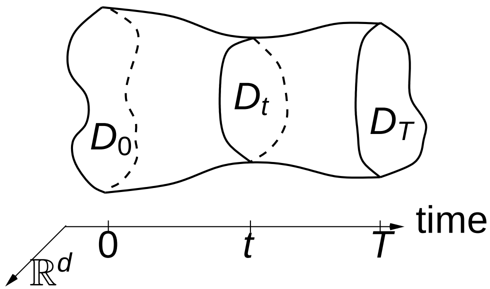

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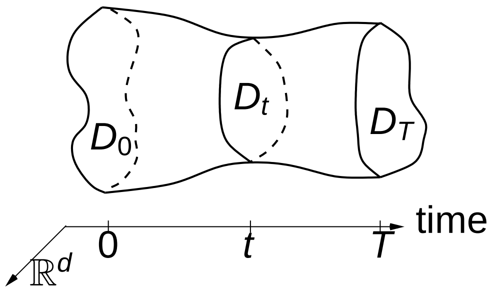

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Under "suitable" assumptions $Q(T, g, f, k, x) = u(0, x)$. Feynman-Kac representation of the PDE

$$\begin{cases} (\partial_t u + L_t u - ku + f)(t, x) = 0, \ (t, x) \in \mathcal{D}, \\ u(t, x) = g(t, x), \ (t, x) \in \mathcal{PD}, \end{cases} \quad \text{(PDE)}$$

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- $L_t\varphi(x) = \langle b(t, x), D_x\varphi(x)\rangle + \frac{1}{2}\mathrm{tr}(a(t, x)D_x^2\varphi(x))$ generator of X_t,
- $\mathcal{PD} := \partial\mathcal{D}\backslash[\{0\} \times D_0]$ parabolic boundary of $\mathcal{D}$,
- k (potential/interest rate), f (source term/dividend), g (final condition/Pay-off) in the heat equation.

Approximation scheme

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Control of the weak error

$$\begin{aligned}
\mathcal{E}_h^D &:= (Q^h - Q)(T,g,f,k,x) = \mathbb{E}_x[\Big\{g(\tau^h \wedge T, X_{\tau^h\wedge T}^h) - g(\tau^h \wedge T, \Pi_{\bar{D}_{\tau^h\wedge T}}(X_{\tau^h\wedge T}^h))\Big\} Z_{\tau^h\wedge T}^h] + \\
&\quad + \mathbb{E}_x[g(\tau^h \wedge T, \Pi_{\bar{D}_{\tau^h\wedge T}}(X_{\tau^h\wedge T}^h))Z_{\tau^h\wedge T}^h + \int_0^{\tau^h\wedge T} Z_{\phi(s)}^h f(\phi(s), X_{\phi(s)}^h)ds] - u(0,x) \\
&:= \mathcal{E}_\Pi^h(T,g,k,x) + \mathbb{E}\left[\sum_{0\leq t_i<\tau^h\wedge T} \Big\{u(t_{i+1}, \Pi_{\bar{D}_{t_{i+1}}}(X_{t_{i+1}}^h))Z_{t_{i+1}}^h - u(t_i, \Pi_{\bar{D}_{t_i}}(X_{t_i}^h))Z_{t_i}^h + Z_{t_i}^h f(t_i, X_{t_i}^h)h\Big\}\right] \\
&\quad + O_{pol}(h) := \mathcal{E}_\Pi^h(T,g,k,x) + \mathcal{E}_{PDE}^h(T,g,f,k,x) + O_{pol}(h).
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- Discrete exit time $\tau^h := \inf\{t_i > 0 : X_{t_i}^h \notin D_{t_i}\}$,
- Approximation of (Q) by :

$$Q^h(T, g, f, k, x) := \mathbb{E}_x[g(\tau^h \wedge T, X_{\tau^h \wedge T}^h) Z_{\tau^h \wedge T}^h + \int_0^{\tau^h \wedge T} Z_{\phi(s)}^h f(\phi(s), X_{\phi(s)}^h) ds],\ Z_t^h = e^{-\int_0^t k(\phi(r), X_{\phi(r)}^h) dr}.$$

Tools

- Regularity of (PDE) Itô-Taylor expansions.
- Control of the overshoot.

Limit Theorem for the Overshoot : Assumptions

Non characteristic boundary condition :

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Let $\mathcal{D} \in C^2$ and $F(t,x) := d(x, \partial D_t)$ (signed distance of x to the boundary of D_t).

(NCB) $\exists r_0 > 0,\ a_0 > 0$ s.t. $\nabla F(t,x) a(t,x) \nabla F(t,x)^* \geq a_0$, $\forall (t,x) \in \bigcup_{0 \leq t \leq T} \{t\} \times V_{\partial D_t}(r_0)$,

where $V_{\partial D_t}(r_0) := \{x \in D_t : d(x, \partial D_t) \leq r_0\}$.

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- **(NCB)** allows intuitively to connect $F(t,X_t) := d(X_t, \partial D_t)$ with a time-changed Brownian motion.
- **(NCB)** guarantees $\mathcal{E}_D^h \underset{h\to 0}{\longrightarrow} 0$.

Limit Theorem for the Overshoot : Statement

Theorem : joint limit laws associated to the overshoot.

Assume $\mathcal{D} \in C^2$, **(NCB)**, $b, \sigma \in C^{(1+\theta)/2,1+\theta}(\bar{\mathcal{D}}),\ \theta \in]0,1[$.

$$(\tau^h, X^h_{\tau^h}, h^{-1/2}F^-(\tau^h, X^h_{\tau^h})) \xrightarrow[h\to 0]{\text{(law)}} (\tau, X_\tau, |\nabla F \sigma(\tau, X_\tau)|Y),$$

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- Distribution function of Y, $H(y) := (\mathbb{E}[\bar{X}_{\bar{\tau}^+}])^{-1} \int_0^y \mathbb{P}[\bar{X}_{\bar{\tau}^+} > z] dz$, $\bar{X}_0 := 0, \forall n \geq 1, \bar{X}_n := \sum_{i=1}^n G^i, (G^i)_{i \geq 0}$ i.i.d. $\mathcal{N}(0,1)$, $\bar{\tau}^+ := \inf\{n \geq 0 : \bar{X}_n > 0\}$.
- Y asymptotic law of the Brownian overshoot, cf. Siegmund (Journal of Applied Proba. 1979).
- $\mathbb{E}(Y) = \frac{\mathbb{E}[\bar{X}^2_{\tau^+}]}{2\mathbb{E}[\bar{X}_{\tau^+}]} := c_0 = -\frac{\zeta(1/2)}{\sqrt{2\pi}} = 0.5826...$, explicit knowledge of the constant important for the correction procedure.

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  - approximation with constant domain and coefficients in a neighbourhood of size h^α of the boundary, $\alpha > 0$.
  - renewal arguments apply outside of a neighbourhood of size $h^{1/2-\varepsilon} < h^\alpha,\ \varepsilon > 0$ (w.r.t. to the characteristic Brownian scale of order $h^{1/2}$).

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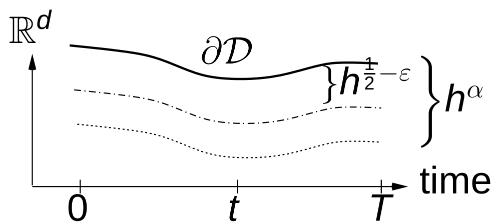

FIG.: The two localization neighbourhoods $\alpha < \frac{1}{2} - \varepsilon$.

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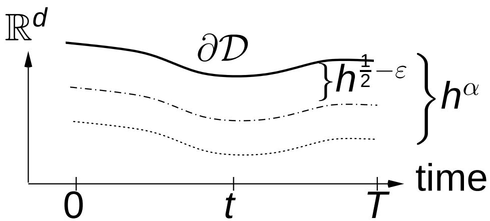

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- Final step : sharp control of the previous approximations w.r.t. the Brownian case and the indicated scales.

Application of the limit theorem for the asymptotics of the overshoot

(B) $\mathcal{D}$, b, σ, g, f, k sufficiently smooth, a uniformly elliptic.

$\rightsquigarrow$ minimal smoothness $u \in C^{(1+\theta)/2,1+\theta}(\bar{\mathcal{D}})$.

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Theorem : error expansion (Gobet, M., SPA 2010)

Assume **(B)**, for h sufficiently small

$$\mathcal{E}_h^D = c_0\sqrt{h}\mathbb{E}_x[\mathbb{I}_{\tau\le T} Z_\tau(\nabla u - \nabla g)(\tau, X_\tau) \cdot \nabla F(\tau, X_\tau)|\nabla F \sigma(\tau, X_\tau)|] + o(\sqrt{h}), \qquad \text{(DEV)}$$

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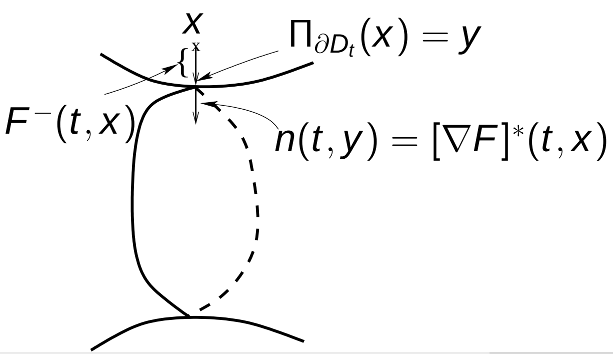

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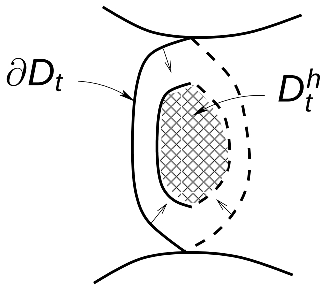
- Error expansion justifies the Romberg extrapolation, i.e. $\sqrt{2}\mathcal{E}_{h/2}^D - \mathcal{E}_h^D = o(\sqrt{h})$.
- Explicit knowledge of c_0 allows to extend the boundary shifting procedure.

Sensitivity w.r.t. the domain

Domain correction (shrinking).

- $\mathcal{D}^h \subset \mathcal{D}$ where $D_t^h = \{x \in D_t : d(x, \partial D_t) > c_0 h^{1/2} |\nabla F \sigma(t,x)|\}$.



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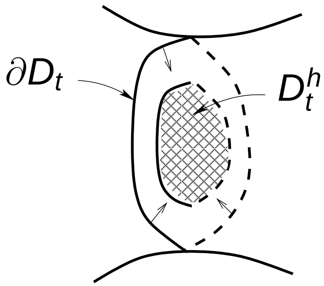

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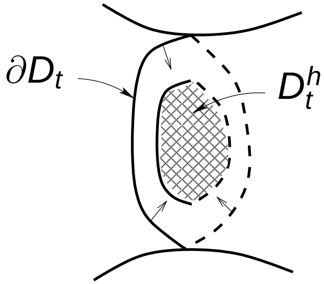

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Theorem : (Gobet, M., SPA (2010))

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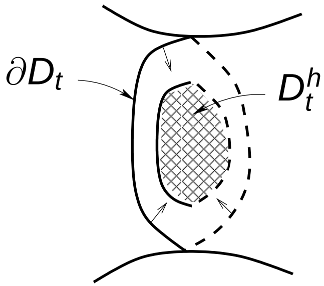

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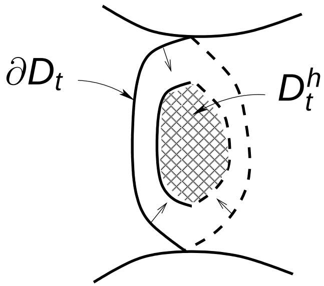

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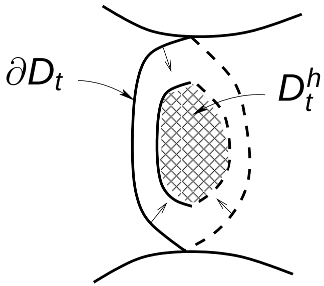

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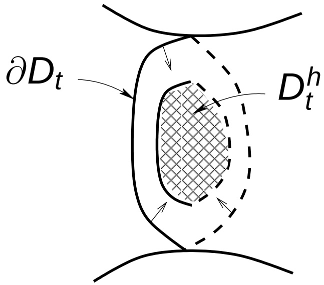

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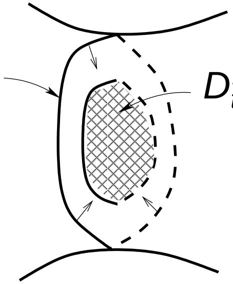

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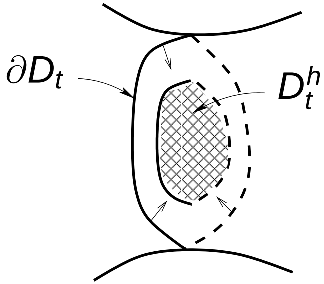

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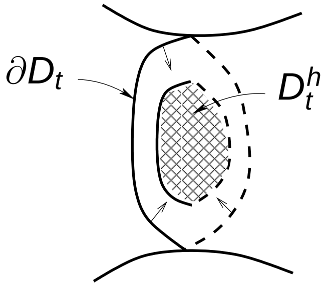

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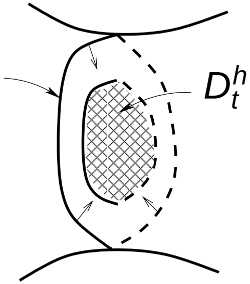

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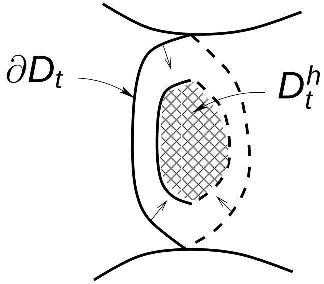

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dX_t &= b(X_t)dt + \sigma(X_t)dW_t,\ \forall x \in \mathbb{R}^3,\ b(x) = (x_2\ x_3\ x_1)^*, \\
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.1	0.169 (199%)	0.0220 (24.4%)
.05	0.114 (133%)	0.0115 (13.1%)
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$$x_0 = (-.7, .3, .7),\ h \in \{.01, .05, .1\},\ \mathrm{Err}_{MC} := \left\{ \frac{1}{MC} \sum_{i=1}^{MC} \left(g(X^{h,i}_{\tau^{h,i}}) + \int_0^{\tau^{h,i}} f(X^{h,i}_{\phi(s)})ds \right) \right\} - u(x_0) :$$

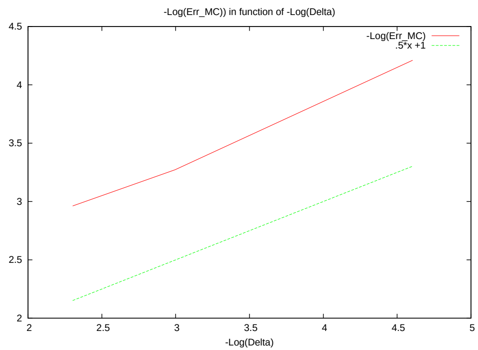

FIG.: Discretization Error induced by the Monte Carlo method (no correction) : $-\log(\mathrm{Err}_{MC})$ in function of $-\log(h)$

Conclusion and extensions

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  - Correction techniques close to the boundary (cf. Borovkov, Peres) in order to get rid of the support condition (digital knock-in/out option w.r.t. the hitting, distribution function of the exit time). Balance between h and the distance of x_0 to the boundary.